Design Methodology for Linear Optimal Control Systems

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I. Introduction

In this Note, we consider the following problem. A linear optimal control system has to be designed for a linear multivariable time-independent plant so as to satisfy practical design specifications. Practical specifications will generally include multiple and conflicting performance objectives.[1] These objectives cannot be mapped easily to the state and control weighting matrices and designers invariably resort to trial-and-error iterations (see Ref. 2). Asymptotic pole-placement/model-matching techniques have been proposed[3,4] and can be used in certain cases to determine the weighting matrices. However, general design specifications are difficult to include. In this work, a design procedure is proposed for solving the above-mentioned problem. This procedure consists of the minimization of a functional in an iterative process, where at each iteration step a linear optimal control problem is solved. This functional incorporates the performance specifications and the linear optimal feedback gains. The effectiveness of the proposed design procedure is demonstrated via an example dealing with the design of a flight control system. The organization of this Note is as follows. In Sec. II, the proposed design methodology is presented. In Sec. III, the application of the proposed method is demonstrated in the design of a decoupled lateral control system for a remotely piloted vehicle (RPV). Different bandwidths are used for controlling the roll angle and the sideslip angle.

II. Proposed Synthesis Methodology

Consider the augmented system consisting of the plant (A_p, B_p, C_p) and the augmented integrators (Fig. 1):

$$\dot{x}(t) = Ax(t) + Bu(t) + B_r r(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where

$$A = \begin{bmatrix} A_p & 0 \\ C_p & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_p \\ 0 \end{bmatrix}$$

$$C = [C_p \quad 0], \qquad B_r = \begin{bmatrix} 0 \\ -I \end{bmatrix}$$

and where $x(t) = [x_p^T(t), x_e^T(t)]^T$ and $x_e(t) = \int_0^t [y(t) - r(t)]\, dt$. In addition, $x(t) \in R^m$ is the system state, $y(t) \in R^n$ is the system controlled output, $u(t) \in R^q$ is the control input, $q \geq n$, $r(t) \in R^n$ is the reference input, $x_p(t) \in R^{m_p}$ is the plant state, and $x_e(t) \in R^n$ is the integrator state. Furthermore, $A \in R^{m \times m}$, $B \in R^{m \times q}$, $C \in R^{n \times m}$, $A_p \in R^{m_p \times m_p}$, $B_p \in R^{m_p \times q}$, $C_p \in R^{n \times m_p}$, 0 denotes the zero matrix of appropriate dimensions, and I denotes the unit matrix of appropriate dimensions. It is assumed that the initial conditions of the system state at $t = 0$ are $x(0) = x_0$.

In this Note, we are interested in computing the feedback gain matrix $F = [F_1 \; F_2]$ for the multivariable proportional plus integral (PI) control system such that the given performance specifications on the system are satisfactorily met.

Consider the following auxiliary problem: Determine the optimal feedback matrix F in the following feedback control law, in the case $r(\cdot) = 0$,

$$u(t) = -Fx(t) = -F_1 x_p(t) - F_2 x_e(t) \tag{3}$$

such that the following quadratic performance index is minimized:

$$I_0 = \frac{1}{2}\int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt \tag{4}$$

In Eq. (4), Q is the state weighting matrix, $Q = Q^T = M^T M \geq 0$, and R is the control weighting matrix, $R = R^T > 0$. Necessary and sufficient conditions for the existence of an optimal PI control law of the form given in Eq. (3) are 1) the pair (A, B) must be completely controllable and 2) the pair (A, M) must be completely observable.[2] If these conditions are satisfied, then the optimal feedback gain F is given by

$$F = R^{-1}B^T P \tag{5}$$

where the matrix $P = P^T$, and P is the unique positive definite solution of the control algebraic Riccati equation:

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{6}$$

The closed-form solution of the auxiliary problem will be used in the sequel in the proposed design procedure.

Using Eqs. (1-3) and the resulting F from Eqs. (5) and (6), the closed loop system is given by

$$\dot{x}(t) = (A - BF)x(t) + B_r r(t), \qquad x(0) = x_0 \tag{7}$$

$$y(t) = Cx(t) \tag{8}$$

In this work, the weighting matrices are chosen to have the following structure:

$$Q = \mathrm{diag}\,(g_1^2, g_2^2, \ldots, g_m^2) \tag{9}$$

$$R = I \tag{10}$$

where $\{g_i\}$ are real numbers and $g = [g_1, g_2, \ldots, g_m]$ becomes the design parameter vector. By selecting Q diagonal and R the unit matrix, the following guaranteed multivariable stability margins are obtained[2]: phase margin $= \pm 60$ deg and gain margin $= [0.5, +\infty)$. The formulation given in Eqs. (9) and (10) has a minimal number of design parameters that is equal to the plant order m. It is also consistent with the fact that the relative and not absolute values of Q and R are important [Eq. (4)].

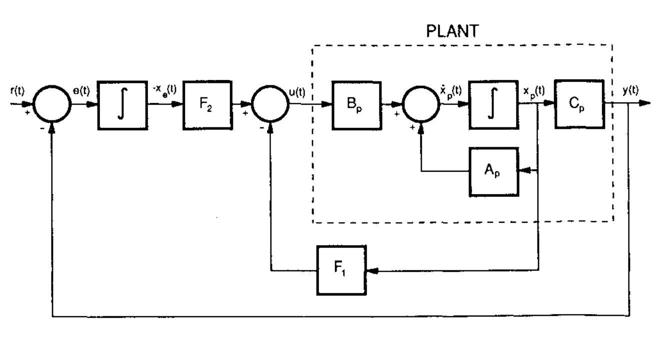

Fig. 1 Block diagram of the proportional plus integral control system.

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However, practical control system design specifications will require the minimization of additional functions. In this work, we consider time-domain decoupled command tracking specifications. We denote by $y^{cj}(t) = [y_1^{cj}(t), \ldots, y_n^{cj}(t)]$ the closed-loop system response to $r^j(t) = (0,0,\ldots,1,\ldots,0)$, $t \ge 0$, where in $r^j(t)$ the jth component is 1 and the rest are 0, $j = 1,\ldots,n$. Also, denote by $y^{d_j}(t) = [y_1^{d_j}(t), \ldots, y_n^{d_j}(t)]$ the desired closed-loop response to $r^j(t)$, $j = 1,\ldots,n$. For calculation purposes, we consider discrete time points $t \in \mathfrak{I}_N = \{t : t = kT, k = 0,1,\ldots,N\}$ and T is the sampling period. The problem addressed here is to calculate the state weighting matrix Q such that the functional J will be minimized subject to Eqs. (5) and (6) and (7) and (8):

$$J = \sum_{i=1}^{q}\sum_{j=1}^{m} c_0(F_{ij})^2 + \sum_{j=1}^{n}\sum_{k=0}^{N}\left[y_j^{cj}(k) - y_j^{d_j}(k)\right]^2 + \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{k=0}^{N}\left[y_i^{cj}(k)\right]^2 \tag{11}$$

Here c_0 is a positive real number, $F = [F_{ij}]$ is the linear optimal feedback gain obtained by Eqs. (5) and (6), and where the notation $y_j^{cj}(kT) = y_j^{cj}(k)$ is being used. Thus, the design procedure proposed in this work consists of the following iterative procedure for the minimization of J:[5]

Step 1: Choose $g^{(0)} = [g_1^{(0)}, \ldots, g_m^{(0)}]$ and numbers h, $\Delta\tau$, $k_{\max}$, δ, μ, δ_1, and ϵ. Set $i = 0$, $j = 2$, $s = 0$, $p = 1$, and $k = -1$.

Step 2: Using Eqs. (5) and (6) and (9) and (10), and solving Eqs. (7) and (8) for $r^j(t)$, $j = 1,\ldots,n$, to obtain $y^{cj}(t)$, $j = 1,\ldots,n$, respectively, $t \in \mathfrak{I}_N$, compute, Eq. (11), $J(g^{(0)})$, $J(g^{(0)} + h\hat{e}_\nu)$, $\nu = 1,\ldots,m$, and $a_0 = -\nabla J(g^{(0)})$, $v_0 = 0.5 a_0 \Delta\tau$ where $[J(g^{(0)})]_\nu \cong [J(g^{(0)} + h\hat{e}_\nu) - J(g^{(0)})]/h$, and $\hat{e}_\nu$ is the unit vector along the νth axis in R^m.

Step 3: Set $k = k + 1$ and compute $\|\Delta g^{(k)}\| = \|v_k\|\Delta\tau$.

Step 4: If $\|\Delta g^{(k)}\| < \delta$, go to step 5; otherwise set $v_k = \delta v_k/(\Delta\tau\|v_k\|)$, and go to step 6.

Step 5: Set $p = p + \delta_1$, $\Delta\tau = p\Delta\tau$.

Step 6: If $s < \mu$, go to step 7; otherwise set $\Delta\tau = \Delta\tau/2$, $g^{(k)} = (g^{(k)} + g^{(k-1)})/2$, $v_k = (v_k + v_{k-1})/4$, $s = 0$, and go to step 7.

Step 7: Set $g^{(k+1)} = g^{(k)} + v_k\Delta\tau$.

Step 8: Compute (in the same manner as in step 2)

$$a_{k+1} = -\nabla J(g^{(k+1)}), \qquad v_{k+1} = v_k + a_{k+1}\Delta\tau$$

Step 9: If $a_{k+1}^T a_k > 0$, set $s = 0$ and go to step 10; otherwise set $s = s + 1$, $p = 1$, and go to step 10.

Step 10: If $\|a_{k+1}\| \le \epsilon$ or $k \ge k_{\max}$, stop; otherwise go to step 11.

Step 11: If $\|v_{k+1}\| > \|v_k\|$, set $i = 0$, and go to step 3; otherwise set $g^{(k+2)} = (g^{(k+1)} + g^{(k)})/2$, $i = i + 1$, and go to step 12.

Step 12: If $i \le j$, set $v_{k+1} = (v_{k+1} + v_k)/4$, set $k = k + 1$, and go to step 8; otherwise set $v_{k+1} = 0$, $j = 1$, $k = k + 1$, and go to step 8.

In this work the following parameter values were used: $\Delta\tau = 0.5$, $\delta = 100$, $\delta_1 = 0.001$, $h = 10^{-5}$, $\mu = 3$, and $\epsilon = 10^{-5}$. For $k_{\max}$ see the example given in this Note. In general, due to lack of convexity the preceding algorithm in steps 1–12 will find only a local minimum.

III. Decoupled Flight Control System Design for an RPV

The application of the proposed method will be demonstrated in the design of a decoupled lateral control system for an RPV. The state space matrices A_p, B_p, and C_p of the lateral airframe plus first-order actuator models were obtained from Ref. 6. The state space model is represented by

$$\dot{x}_p(t) = A_p x_p(t) + B_p u(t) = \begin{bmatrix} -8.527\text{E-}2 & -1.423\text{E-}4 & -0.9994 & 4.142\text{E-}2 & 0 & 0.1862 \\ -46.86 & -2.757 & 0.3896 & 0 & -124.3 & 128.6 \\ -0.4248 & -6.224\text{E-}2 & -6.714\text{E-}2 & 0 & -8.792 & -20.46 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix} u(t)$$

$$y(t) = C_p x_p(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_p(t)$$

The state vector is $x_p(t) = [\beta(t), p(t), r(t), \phi(t), de(t), dr(t)]^T$ (sideslip angle in radians, roll rate in radians/second, yaw rate in radians/second, roll angle in radians, elevon surface deflection in radians, rudder surface deflection in radians). The control input is $u(t) = [\text{dec}(t), \text{drc}(t)]^T$ (elevon servo command and rudder servo command in radians). The outputs to be controlled are $y(t) = [\phi(t), \beta(t)]^T$ (roll angle and sideslip angle in radians). The multivariable PI control system in Fig. 1 is used and the reference input is $r(t) = [\phi_c(t), \beta_c(t)]^T$. The objective is to independently control the roll angle and the sideslip angle. Using the notation of Sec. II, for $(\phi_c(t), \beta_c(t)) = (1 \text{ rad}, 0)$, the closed-loop response $(\phi(t), \beta(t))$ must be approximately equal to $(y_1^{d_1}(t), 0)$ for $t \in \mathfrak{I}_N$, where $y_1^{d_1}(t)$ is the step response of a second-order system with a natural frequency of 4 rad/s and a damping ratio of 0.7 and $y_1^{d_1}(t)$ is shown in Fig. 2a. For $(\phi_c(t), \beta_c(t)) = (0, 1 \text{ rad})$, the closed-loop response $(\phi(t), \beta(t))$ must be approximately equal to $(0, y_2^{d_2}(t))$ for $t \in \mathfrak{I}_N$, where $y_2^{d_2}(t)$ is the step response of a second-order system with a natural frequency of 2 rad/s and a damping ratio of 0.7 and $y_2^{d_2}(t)$ is shown in Fig. 2c. As $y_1^{d_1}(t)$ and $y_2^{d_2}(t)$ have different rise times, the roll angle and the sideslip angle will in effect be controlled independently with different bandwidths. This is an important requirement for practical multivariable systems.[7]

The algorithm described by steps 1–12 in Sec. II was applied until convergence to a practical and acceptable solution was obtained. The following numerical data was used: $T = 0.05$ s, time horizon $= NT = 5$ s, i.e., $N = 100$, $c_0 = 10^{-4}$, Q_0 = initial value of state weighting matrix = diag (1, 1, 1, 1, 1, 1, 1, 1) and x_0 = initial conditions of the augmented system state (at $t = 0$) = $[0, 0, 0, 0, 0, 0, 0, 0]^T$. After 200 iterations the following state weighting matrix was obtained: Q = diag (0.88759, 0.13986, 8.0919E-2, 0.42166, 1.2210E-5, 5.2586, 61.079, 9.2040). The optimal feedback gain F turned out to be

$$F = \begin{bmatrix} 2.0025 & -0.36386 & -0.43114 & -2.3539 & 1.3607 & -0.57350 & -6.4614 & 1.7067 \\ 1.4177 & 0.25438 & -0.58216 & 1.6123 & -0.57350 & 2.2278 & 4.3964 & 2.5083 \end{bmatrix}$$

For a unit-step roll angle command, the response of the controlled variables and control surface deflections are shown in Figs. 2a and 2b, respectively. For a unit-step sideslip angle command, the corresponding responses are shown in Figs. 2c and 2d, respectively. It is evident from Figs. 2a and 2c that the

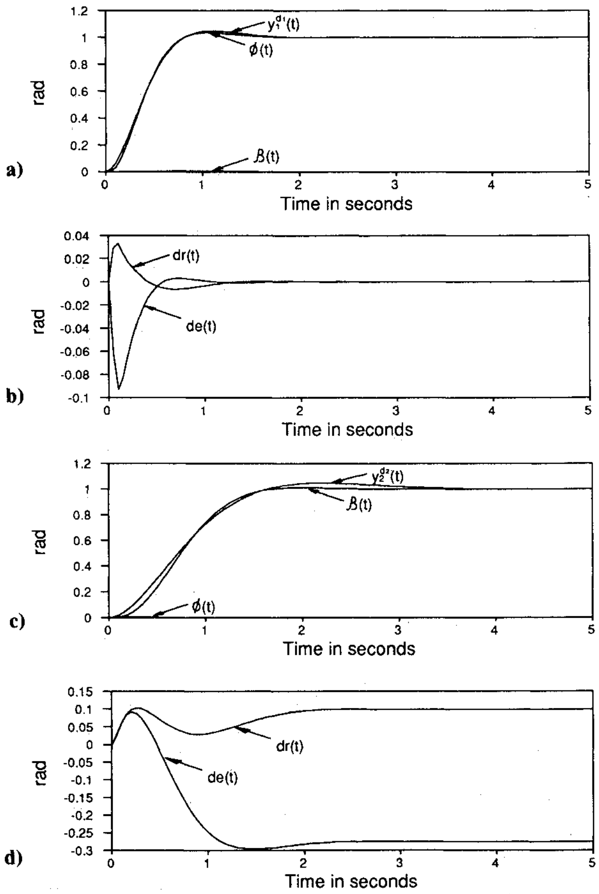

Fig. 2 Reference command $(\phi_c(t), \beta_c(t)) = (1\ \mathrm{rad}, 0)$: a) controlled variables and b) control surface deflections; reference command $(\phi_c(t), \beta_c(t)) = (0, 1\ \mathrm{rad})$: c) controlled variables and d) control surface deflections.

specifications are approximately met. The roll-angle tracking maneuver with approximately zero sideslip shown in Fig. 2a is practically used in RPV flight control. A PC-AT compatible computer requires approximately 60 min to obtain the preceding solution. By including numeric and step-response graphics output at each iteration, the proposed design procedure can provide useful visual information on how the actual step response of the system converges to the desired step response.

IV. Conclusion

This Note has proposed a synthesis methodology for automating the design of linear optimal control systems such that practical performance specifications are met in a satisfactory manner. Decoupled command tracking specifications in the time domain have been considered at present and a multivariable proportional plus integral control structure has been employed. The proposed design procedure has been applied in the design of a decoupled lateral control system for an RPV. The results obtained, part of which are presented in Fig. 2, demonstrate the effectiveness of the proposed design method.

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Model Reduction for Systems with Integrators

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Introduction

Model order reduction methods for stable linear systems are based on joint controllability and observability tests through balancing of system grammians.[1] For systems with integrators, grammians do not exist; thus, model reduction based on its grammian properties cannot be executed. Note, however, that systems with integrators are controllable and observable; hence, these properties can still be used for model reduction. In this Note the reduction algorithm for systems with integrators is derived.

Reduction

Consider a linear system with the state-space representation (A, B, C), with p inputs, q outputs, and n states. It is called a system with integrators, if it is observable and controllable, has $n - m$ poles stable, has the remaining m poles at zero, and has a nondefective A (geometric multiplicity of poles at zero is m). The reduction problem for systems with integrators is solved by introducing antigrammians.

For a controllable and observable triple (A, B, C) the matrices V_c, V_o satisfying the following Riccati equations:

$$V_cA + A^TV_c + V_cBB^TV_c = 0$$
$$V_oA^T + AV_o + V_oC^TCV_o = 0 \tag{1a}$$

are the controllability and observability antigrammians. For stable controllable and observable systems, $V_c = W_c^{-1}$, $V_o = W_o^{-1}$, where W_c, W_o are controllability and observability grammians satisfying the Lyapunov equations

$$AW_c + W_cA^T + BB^T = 0, \qquad W_oA + A^TW_o + C^TC = 0 \tag{1b}$$

The grammians for a system with integrators do not exist, but antigrammians do; for an unobservable or uncontrollable sys-

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